Mathematical analysis of the transient loads for a deformed joint or welding in a railway track

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ABSTRACT

The railway track is modeled as a continuous beam on elastic support. Train circulation is a random dynamic phenomenon and, according to the different frequencies of the loads it imposes, there exists the corresponding response of track superstructure. At the moment when an axle passes from the location of a sleeper, a random dynamic load is applied on the sleeper. The theoretical approach for the estimation of the dynamic loading of a sleeper demands the analysis of the total load acting on the sleeper to individual component loads-actions.

The dynamic component of the load of the track depends on the mechanical properties (stiffness, damping) of the system ”‘vehicle-track”’, and on the excitation caused by the vehicle's motion on the track. The response of the track to the aforementioned excitation results in the increase of the static loads on the superstructure. The dynamic load is primarily caused by the motion of the vehicle's Non-Suspended (Unsprung) Masses, which are excited by track geometry defects, and, to a smaller degree, by the effect of the Suspended (Sprung) Masses. In order to formulate the theoretical equations for the calculation of the dynamic component of the load, the statistical probability of exceeding the calculated load -in real conditions- should be considered, so that the corresponding equations refer to the standard deviation (variance) of the load.

In the present paper, the dynamic component is investigated through the second order differential equation of motion of the Non Suspended Masses of the Vehicle and specifically the transient response of the reaction/action on each support point (sleeper) of the rail. The case of a deformed or bent joint or welding is analyzed through the second order differential equation of motion and the solution is investigated.

KEYWORDS

Static Stiffness Coefficient — Loads — Non-Suspended Masses — Rail — Ballast — Pad — Sleeper — Joint — Welding.

1. Introduction

The railway track is modeled as a continuous beam on elastic support. Train circulation is a random dynamic phenomenon and, according to the different frequencies of the loads it im-poses, there exists the corresponding response of track superstructure. At the moment when an axle passes from the location of a sleeper, a random dynamic load is applied on the sleeper. The theoretical approach for the estimation of the dynamic loading of a sleeper demands the analysis of the total load acting on the sleeper to individual component loads-actions, which,

in general, can be divided into:

- the static component of the load, and the relevant to it reaction/action per support point of the rail (sleeper).

- the dynamic component of the load, and the relevant to it reaction/action per support point of the rail.

The static component of the load on a sleeper, in the classical sense, is the load undertaken by the sleeper when a vehicle axle at standstill is situated exactly above the location of the sleeper. At low frequencies, however, the load is essentially static. The static load is further analyzed into individual component loads: the static reaction/action on a sleeper due to wheel load and the semi-static reaction/action due to cant deficiency [1].

The dynamic component of the load of the track depends on the mechanical properties (stiffness, damping) of the system "vehicle-track", and on the excitation caused by the vehicle's motion on the track (Figure 1). The response of the track to the aforementioned excitation results in the increase of the static loads on the superstructure. The dynamic load is primarily caused by the motion of the vehicle's Non-Suspended (Unsprung) Masses, which are excited by track geometry defects, and, to a smaller degree, by the effect of the Suspended (sprung) Masses. The Non Suspended Masses of a Vehicle are situated below the primary suspension of the vehicle. This means that the axle with the wheels plus a percentage of a semi-suspended electric motor, in the case of locomotives, belong to them. All the rest are the Suspended Masses of the vehicle. To the Non Suspended Masses of the vehicle a portion of the track-mass is added during their motion. In order to formulate the theoretical equations for the calculation of the dynamic component of the load, the statistical probability of exceeding the calculated load -in real conditions- should be considered, so that the corresponding equations refer to the standard deviation (variance) of the load [1, 2].
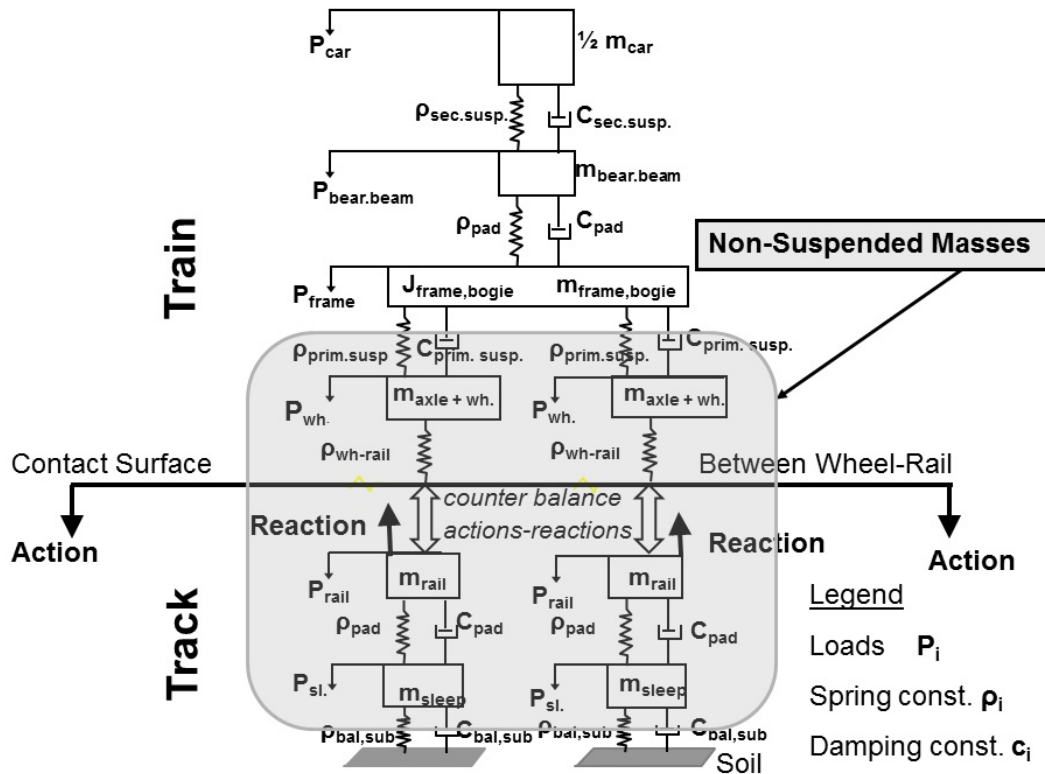

Figure 1. *The system "Vehicle-Track" as an Ensemble of Springs and Dashpots; over the contact surface the vehicle, below the contact surface the track*

2. Static Component of the Load

2.1 Static Reaction/Action on a Rail Support Point due to the Static Wheel Load

The most widely used theory (referred to as the Zimmermann theory or formula) examines the track as a continuous beam on elastic support whose behavior is governed by the following equation [3]:

$$\frac{d^4y}{dx^4} = -\frac{1}{E \cdot J} \cdot \frac{d^2M}{dx^2} \tag{1}$$

where y is the deflection of the rail, M is the moment that stresses the beam, J is the moment of inertia of the rail, and E is the modulus of elasticity of the rail.

From the formula above it is derived that the reaction at each support point of the rail (that is of a sleeper) R_{static} is:

$$R_{stat} = \frac{Q_{wheel}}{2\sqrt{2}} \cdot \sqrt[4]{\frac{\ell^3 \cdot \rho}{E \cdot J}} \Rightarrow \frac{R_{stat}}{Q_{wheel}} = \bar{A} = \bar{A}_{stat} = \frac{1}{2\sqrt{2}} \cdot \sqrt[4]{\frac{\ell^3 \cdot \rho}{E \cdot J}} \tag{2}$$

where Q_{wheel} the static wheel load, ℓ the distance among the sleepers, E and J the modulus of elasticity and the moment of inertia of the rail, R_{stat} the static reaction/action on the sleeper, ρ (or c in German literature) reaction coefficient of the sleeper which is defined as: $\rho = R/y$, and is a quasi-coefficient of track elasticity (stiffness) or a spring constant of the track, $\bar{A} = \bar{A}stat$ equals to R_{stat}/Q_{wheel}, that is the percentage of the acting (static) load of the wheel that the sleeper undertakes as (static) reaction.

In reality, the track consists of a sequence of materials –in the vertical axis– (substructure, ballast, sleeper, elastic pad/ fastening, rail), that are characterized by their individual coeffi-cients of elasticity (static stiffness coefficients) ρ_i (Figure 2). Hence, for each material:

$$\rho_i = \frac{R}{y_i} \Rightarrow y_i = \frac{R}{\rho_i} \Rightarrow y_{total} = \sum_{i=1}^{\nu} y_i = \sum_{i=1}^{\nu} \frac{R}{\rho_i} = R \cdot \sum_{i=1}^{\nu} \frac{1}{\rho_i} \Rightarrow \frac{1}{\rho_{total}} = \sum_{i=1}^{\nu} \frac{1}{\rho_i} \tag{3}$$

where ν is the number of various layers of materials that exist under the rail -including rail– elastic pad, sleeper, ballast etc.

2.2 Semi-static Reaction/Action on a Rail Support Point due to Superelevation Deficiency

This load is produced by the centrifugal acceleration exerted on the wheels of a vehicle that is running in a curve with cant deficiency. Cant deficiency or unbalanced superelevation/cant [4, p. 604] is defined as the difference (deficit or excess in mm) of the designed superelevation in a curve from the theoretic one that is needed to fully counterbalance the centrifugal acceleration in the cross section of a track on a curve. It is not, however, a dynamic load in the sense of the load referred to in the next paragraph. Therefore, it is often considered to be a semi-static load. The following equation [1, 5, 6]:

$$Q_\alpha = \frac{2 \cdot \alpha \cdot h_{CG}}{e^2} \cdot Q_{wheel} \tag{4}$$

provides the increase Q_α of the vertical static load Q_{wheel} of the wheel, at curves with cant deficiency. In the above equation α is the cant deficiency, h_{CG} the height of the center of gravity of the vehicle from the rail head, e the track gauge. The semi-static reaction of the sleeper is:

$$R_{semi-stat} = \bar{A} \cdot Q_\alpha R_{semi-stat} = \bar{A} \cdot Q_\alpha \tag{5}$$

and the total static reaction is:

$$R_{stat-total} = R_{stat} + R_{semi-stat} R_{stat-total} = R_{stat} + R_{semi-stat} \tag{6}$$

3. Dynamic Component of the Load

3.1 The Non Suspended Masses : General Form of the Second Order Differential Equation of Motion in a Railway Line

The Suspended (sprung) Masses of the vehicle –masses situated above the primary suspension (Figure 1)– create forces with very small influence on the wheel's trajectory and on the system's excitation. This enables the
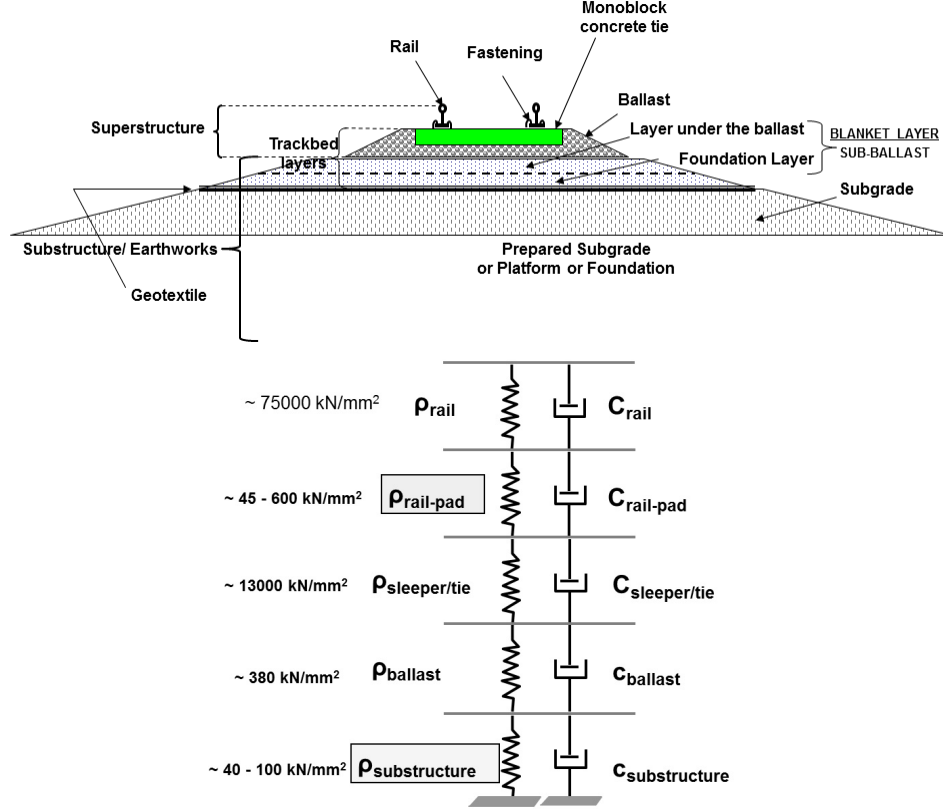
Figure 2. *The Cross-section of Ballasted Track and Characteristic Values of the Static Stiffness Coefficients.*

simulation of the track as an elastic media with damping as shown in Figure 3, depicting the rolling wheel on the rail running table [7]. Forced oscillation is caused by the irregularities of the rail running table (like an input random signal) –which are represented by n–, in a gravitational field with acceleration g. There are two suspensions on the vehicle for passenger comfort purposes: primary and secondary suspension. Moreover, a section of the mass of the railway track participates in the motion of the Non-Suspended (Unsprung) Masses of the vehicle. These Masses are situated under the primary suspension of the vehicle.

If the random excitation (track irregularities) is given, it is difficult to derive the response, unless the system is linear and invariable. In this case the input signal can be defined by its spectral density and from this we can calculate the spectral density of the response. The theoretical results confirm and explain the experimental verifications ([6], p.39, 71).

The equation for the interaction between the vehicle's axle and the track becomes [3, 8]:

$$(m_{NSM} + m_{TRACK}) \cdot \frac{d^2 y}{dt^2} + \Gamma \cdot \frac{dy}{dt} + h_{TRACK} \cdot y = -m_{NSM} \cdot \frac{d^2 n}{dt^2} + (m_{NSM} + m_{SM}) \cdot g \tag{7}$$

where m_{NSM} the Non-Suspended Masses of the vehicle, m_{TRACK} the mass of the track that participates in the motion, m_{SM} the Suspended Masses of the vehicle that are cited above the primary suspension of the vehicle, Γ damping constant of the track, h_{TRACK} the total dynamic stiffness coefficient of the track (for its calculation see [8]), n the fault ordinate of the rail running table and y the total deflection of the track.

The phenomena of the wheel-rail contact and of the wheel hunting, particularly the equivalent conicity of the wheel and the forces of pseudo-glide, are non-linear. In any case the use of the linear system's approach is valid for speeds lower than the $V_c ritical$ $500 km/h$. The integration for the non-linear model (wheel-rail contact, wheel-hunting and pseudoglide forces) is performed through the Runge Kutta method ([6], p.94-95, 80, see also [9], p.171, 351).

The solution of this second order differential equation of motion (forced damped vibration) gives the increase
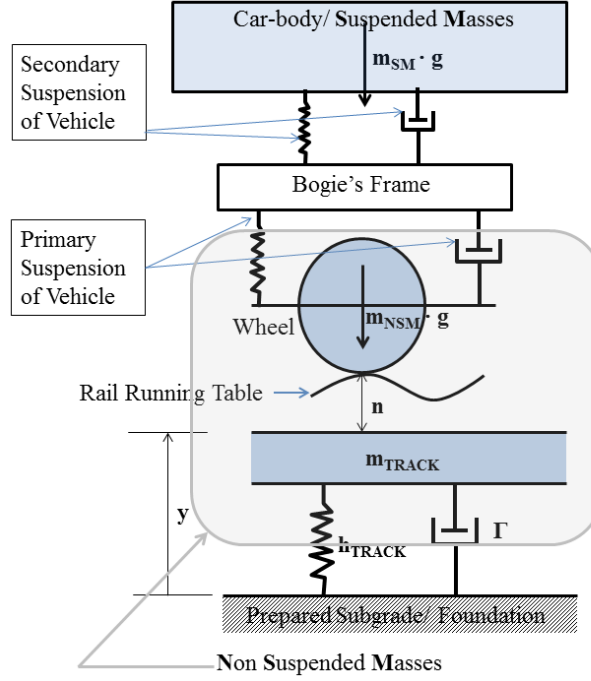
Figure 3. *Model of the Non-Suspended Masses rolling on the rail running table/surface.*

of the $R_{stat-total}$ of equation (6), by the dynamic component of the Load due to the Non Suspended and the Suspended Masses of the Vehicle, mainly based on the steady-state solution. The solution for the dynamic component due to the Non Suspended Masses and its verification through measurements is cited in [10]. The solution for the Suspended Masses is cited in [10, 11]. In the next paragraphs we investigate the transient component of the general solution of the equation (7).

3.2 Railway Track's Defect of Cosine Form

The theoretical analysis for the additional –to the static and semi-static component– dynamic component of the load due to the Non Suspended Masses and the Suspended Masses of the vehicle, lead to the examination of the influence of the Non Suspended Masses only, since the frequency of oscillation of the Suspended Masses is much smaller than the frequency of the Non Suspended Masses. If m_{NSM} represents the Non Suspended Mass, m_{SM} the Suspended Mass and m_{TRACK} the Track Mass participating in the motion of the Non Suspended Masses of the vehicle, the differential equation is:

$$m_{NSM} \cdot \frac{d^2z}{dt^2} + h_{TRACK} \cdot z = m_{NSM} \cdot g \tag{8a}$$

where g the acceleration of gravity and the dynamic track stiffness coefficient h_{TRACK}:

$$h_{TRACK} = 2\sqrt{2} \cdot \sqrt[4]{\frac{EJ\rho_{total}^3}{\ell^3}} \tag{8}$$

ρ_{total} the total static stiffness coefficient of the track, ℓ the distance among the sleepers, E, J the modulus of elasticity and the moment of inertia of the rail.
The theoretic calculation of m_{TRACK} gives as result [10, 11]:

$$m_{TRACK} = 2\sqrt{2} \cdot m_0 \cdot \sqrt[4]{\frac{EJ\ell}{\rho_{total}}} \tag{9}$$

The equation (8a) is transformed:

$$(m_{NSM} + m_{TRACK}) \cdot \frac{d^2z}{dt^2} + h_{TRACK} \cdot z = m_{NSM} \cdot g \tag{8b}$$

For a comparison of the theoretical track mass to measurements' results see [10, 11]. The particular solution of the differential equation (8b) corresponds to the static action of the weight of the wheel:

$$z = \frac{m_{TRACK} \cdot g}{h_{TRACK}} \tag{11a}$$

Let's suppose that the rolling wheel run over an isolated sinusoidal defect of length λ of the form:

$$n = \frac{a}{2} \cdot \left(1 - \cos\frac{2\pi x}{\lambda}\right) = \frac{a}{2} \cdot \left(1 - \cos\frac{2\pi V t}{\lambda}\right) \tag{11b}$$

where n is the ordinate of the defect, consequently the ordinate of the center of inertia of the wheel is $n + z$. If we name τ_1 the time needed for the overpassing of the defect by the wheel rolling at a speed V:

$$\tau_1 = \frac{\lambda}{V} \tag{10}$$

The differential equation of the motion of the wheel is:

$$m_{NSM} \cdot \frac{d^2}{dt^2}(z+n) + m_{TRACK} \cdot \frac{d^2 z}{dt^2} + h_{TRACK} \cdot z = 0 \Rightarrow$$

$$(m_{NSM} + m_{TRACK}) \cdot \frac{d^2 z}{dt^2} + h_{TRACK} \cdot z = -m_{NSM} \cdot \frac{d^2 n}{dt^2} \Rightarrow \tag{11}$$

$$(m_{NSM} + m_{TRACK}) \cdot \frac{d^2 z}{dt^2} + h_{TRACK} \cdot z = -m_{NSM} \cdot \frac{2a\pi^2}{\tau_1^2} \cdot \cos\frac{2\pi t}{\tau_1}$$

3.3 Passing from a Defect of Cosine Form to a Deformed Joint or Welding

The joints during their Life-Cycle are battered and consequently the rail edges present de-formations and bends. The weldings, in the Continuously Welded Rails (CWR), due to non-correct execution of the welding procedure (mainly poor alignment) or "softer" material in the area of welding, could present also the same image. In Figure 4a a wheel passes a deformed, bent joint (or welding). We can approach the matter beginning with a discontinuity of the rail running table –a change in the inclination of the rail running table along the track– in the form of one angle (as in Figure 4b-upper illustration), instead of two parabolic arcs (as in Figure 4a). We use the "mass-spring-damper" model as depicted in Figure 3.
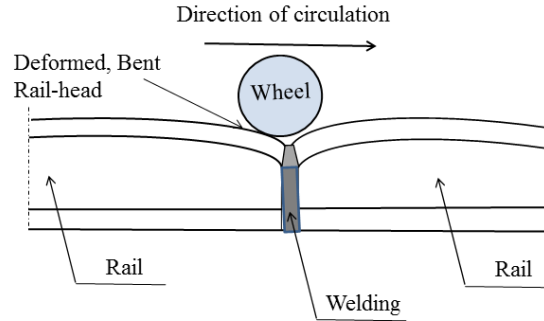

Figure 4a. *Wheel passing a deformed Joint or Welding.*

The equation of the form of the defect is:

$$n = -\alpha \cdot x = -\alpha \cdot V \cdot t \tag{12}$$

where α is the angle in rad and V the speed, for $x > 0$ or $t > 0$.

At this point we have to remember the delta (or Dirac) function $\delta(x)$ and the unit step (Heaviside's) function $H(t)$. The delta function is usually defined as follows ([12, p. 270] and [13, p. 74]):

$$\delta(t) = 0 \;, \quad for \;\; t \neq 0 \;, \quad and \quad \int_{-\infty}^{+\infty} \delta(t) \cdot dt \;\; = 1 \tag{13}$$
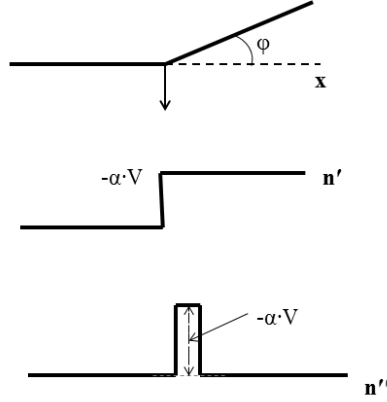
Figure 4b. *Rail defect in the form of an angle in the rail running table/surface (upper illustration), its first derivative (middle illustration) and the second derivative (lower illustration).*

The unit step function ([12], p. 38] and [13, p. 61]) is defined:

$$H(t) = \frac{1}{2} + \frac{1}{2} \cdot sgnt = \begin{cases} 0 & if \quad t < 0 \\ (\frac{1}{2} & if \quad t = 0 \ ) \\ 1 & if \quad t > 0 \end{cases} \tag{14}$$

where the sign function is defined ([13, p. 65]):

$$sgnt = \begin{cases} -1 & for \quad t < 0 \\ +1 & for \quad t > 0 \end{cases} \tag{15}$$

The unit step function of Heaviside is depicted in Figure 5 and comparing Figure 5 to Figure 4b-middle, we conclude that they have similar form.
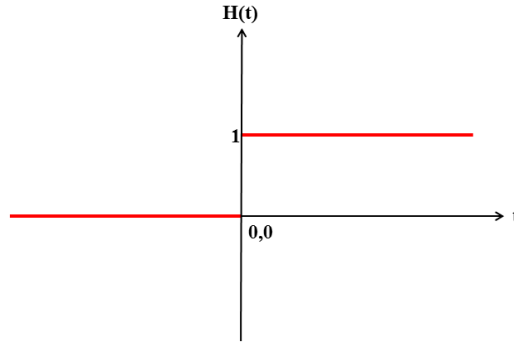

Figure 5. *The unit step function (of Heaviside).*

Differentiating in relation to time *t* the equations (8a, 8b) we can derive:

$$n' = -\alpha \cdot V = -\alpha \cdot V \cdot H(t) \tag{16}$$

In Figure 4b-middle the first derivative n' is depicted. From the properties of the delta function and the unit step function we know that ([13], p. 98, and [14], p. 42), the first derivative of the unit step function $H'(t)$, is the Dirac's delta function $\delta(t)$, consequently:

$$n'' = -\alpha \cdot V \cdot H'(t) = -\alpha \cdot V \cdot \delta(t) \tag{17}$$

In Figure 4b-lower illustration the second derivative n'' is depicted. From the equations (8a, 8b), replacing the second term of the forcing external load due to the angle on the rail running table, and adding the term for damping, we derive:

$$(m_{NSM} + m_{TRACK}) \cdot \frac{d^2 z}{dt^2} + \Gamma \cdot \frac{dy}{dt} + h_{TRACK} \cdot z = -m_{NSM} \cdot \frac{d^2 n}{dt^2} = -m_{NSM} \cdot (-\alpha \cdot V \cdot \delta(t)) \Rightarrow$$

$$\frac{d^2 z}{dt^2} + \frac{\Gamma}{(m_{NSM} + m_{TRACK})} \cdot \frac{dy}{dt} + \frac{h_{TRACK}}{(m_{NSM} + m_{TRACK})} \cdot z = = \frac{m_{NSM}}{\underbrace{(m_{NSM} + m_{TRACK})}_{\approx 1}} \cdot \alpha \cdot V \cdot \delta(t) \Rightarrow \tag{18}$$

$$\frac{d^2 z}{dt^2} + 2 \cdot \zeta \cdot \omega_n \cdot \frac{dy}{dt} + \omega_n^2 \cdot z \approx \alpha \cdot V \cdot \delta(t)$$

3.4 Investigating the Solution of the Second Order Differential Equation of Motion

For the free oscillation (without external force) the equation is:

$$m \cdot \ddot{z} + k \cdot z = 0 \Rightarrow \ddot{z} + \frac{k}{m} \cdot z = 0 \Rightarrow \ddot{z} + \omega_n^2 \cdot z = 0 \tag{19}$$

The general solution is [3]:

$$z(t) = A \cdot \cos(\omega_n t) + B \cdot \sin(\omega_n t) = z(0) \cdot \cos(\omega_n t) + \frac{\dot{z}(0)}{\omega_n} \cdot \sin(\omega_n t) \tag{20}$$

where:

$$A = z(0), \quad B = \frac{\dot{z}(0)}{\omega_n} \tag{21}$$

If we pass to the damped harmonic oscillation of the form:

$$m \cdot \ddot{z} + c \cdot \dot{z} + k \cdot z = p_0 \cdot \cos(\omega t) \Rightarrow$$

$$\ddot{z} + \frac{c}{m} \cdot \dot{z} + \omega_n^2 \cdot z = \frac{p_0}{m} \cdot \cos(\omega t) \Rightarrow \tag{22}$$

$$\ddot{z} + 2 \cdot \zeta \cdot \omega_n \cdot \dot{z} + \omega_n^2 \cdot z = \omega_n^2 \cdot \frac{p_0}{k} \cdot \cos(\omega t)$$

where:

$$\omega_n^2 = \frac{k}{m} \Rightarrow m = \frac{k}{\omega_n^2} \tag{23}$$

The particular solution of the linear second order differential equation (22) is of the form:

$$z_p(t) = C \cdot \sin(\omega t) + D \cdot \cos(\omega t) \Rightarrow$$

$$\dot{z}_p(t) = \omega \cdot C \cdot \cos(\omega t) - \omega \cdot D \cdot \sin(\omega t) \Rightarrow \tag{24}$$

$$\ddot{z}_p(t) = -\omega^2 \cdot C \cdot \sin(\omega t) - \omega^2 \cdot D \cdot \sin(\omega t)$$

Substituting eq. (24) to eq. (22) and after the mathematical procedure we derive ([3], p.110, [15]):

$$\left[(\omega_n^2 - \omega^2) C - 2\zeta \omega_n \omega D \right] \cdot \sin(\omega t) + \left[2\zeta \omega_n \omega C - (\omega_n^2 - \omega^2) D \right] \cdot \cos(\omega t) = \omega_n^2 \frac{p_0}{k} \sin(\omega t) \tag{25}$$

For the equation (25) to be valid for every t, the coefficients of the sine and cosine terms of the equation must be equal and finally solving a two equations system, we derive:

$$C = \frac{p_0}{k} \cdot \frac{\left(\frac{\omega}{\omega_n} \right)^2}{\left[1 - \left(\frac{\omega}{\omega_n} \right)^2 \right]^2 + \left[2\zeta \left(\frac{\omega}{\omega_n} \right) \right]^2}, \tag{28a}$$

$$D = \frac{p_0}{k} \cdot \frac{-2\zeta \left(\frac{\omega}{\omega_n}\right)^2}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta \left(\frac{\omega}{\omega_n}\right)\right]^2} \tag{28b}$$

The complete solution, for the equation (22), is the addition of the solution (20) and of the solution of the equation (22) combined with equation (28):

$$z(t) = \underbrace{e^{-\zeta\omega_n t} \cdot (A \cdot \cos(\omega_D t) + B \cdot \sin(\omega_D t))}_{transient-term} + \underbrace{C \cdot \sin(\omega t) + D \cdot \cos(\omega t)}_{steady-state-term} \tag{29a}$$

where:

$$\omega_D = \omega_n \sqrt{1 - \zeta^2} \omega_D = \omega_n \sqrt{1 - \zeta^2} \tag{29b}$$

In the case of equation (18), we have a constant external force and $\omega = 0$, consequently $sin(\omega t) = 0$ and $D = 0$. There is no steady state term in the solution, but only transient term. The equation (29a) is transformed to:

$$z(t) = \underbrace{e^{-\zeta\omega_n t} \cdot (A \cdot \cos(\omega_D t) + B \cdot \sin(\omega_D t))}_{transient-term} \tag{26}$$

Equation (26) can be written (choosing appropriately the sine form function and not the cosine, since for $t=0$ the value of $z=0$) also in the form of polar coordinates ([16], p.28 and [17], p.22, 24):

$$z(t) = r \cdot \cos(\omega_D t + \theta) \cdot e^{-\zeta\omega_n t} \tag{31a}$$

$$r = \left[z(0)^2 + \left(\frac{\dot{z}(0) + z(0) \cdot \zeta \cdot \omega_n}{\omega_D}\right)^2\right]^{1/2} \tag{31b}$$

$$\theta = -\tan^{-1}\left(\frac{\dot{z}(0) + z(0) \cdot \zeta \cdot \omega_n}{\omega_D \cdot z(0)}\right) \tag{31c}$$

where $p_0 = \alpha V$, $\theta = 0$ since there is no phase difference between the external force and the eigenfrequency. We have for $t = 0$, then $z(0) = 0$ as depicted in Figure 4:

$$z(t) = \left[(0)^2 + \left(\frac{\alpha V + (0)\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}}\right)^2\right]^{1/2} \cdot \cos\left(\omega_n\sqrt{1-\zeta^2} \cdot t\right) \cdot e^{-\zeta\omega_n t} =$$

$$= \frac{\alpha V}{\omega_n\sqrt{1-\zeta^2}} \cdot e^{-\zeta\omega_n t} \cdot \cos\left(\omega_n\sqrt{1-\zeta^2} \cdot t\right) \tag{27}$$

Since the action and the deflection take simultaneously their maximum values at the support point of the rail (sleeper), then the maximum increase of the total action/ reaction, due to the dynamic component owed to the defect, is observed for:

$$\omega_n\sqrt{1-\zeta^2} \cdot t = \pi \Rightarrow t = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \tag{28}$$

at a remote point -from the defect's peak- and so more remote as the ω_n is small this means that in cases of very soft prepared subgrade (or platform).
The dynamic increase of the load, due to a deformed, bent joint or welding is equal to:

$$Q_{dynamic} = \left[\frac{\alpha V}{\omega_n\sqrt{1-\zeta^2}} \cdot e^{-\zeta\omega_n \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}} \cdot \cos(\pi)\right] \cdot h_{TRACK} =$$

$$= \frac{\alpha V h_{TRACK}}{\omega_n\sqrt{1-\zeta^2}} \cdot e^{-\zeta\frac{\pi}{\sqrt{1-\zeta^2}}} \tag{29}$$

Since: $\omega_n = \sqrt{\frac{h_{TRACK}}{m_{NSM}}}$, equation (29) is transformed:

$$Q_{dynamic} = \frac{\alpha V h_{TRACK}}{\sqrt{\frac{h_{TRACK}}{m_{NSM}}} \cdot \sqrt{1-\zeta^2}} \cdot e^{-\zeta \frac{\pi}{\sqrt{1-\zeta^2}}} = k \cdot \alpha \cdot V \cdot \sqrt{m_{NSM} \cdot h_{TRACK}} \qquad (30)$$

where $k = \frac{e^{-\zeta \frac{\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}}$.

The dynamic increase of the load is proportional to the speed V and to the square root of the product of the Non Suspended Mass m_{NSM} times the dynamic stiffness coefficient of track h_{TRACK}. Furthermore the dynamic component of the load due to a deformed, bent joint or welding, $Q_{dynamic}$ decreases when the damping coefficient ζ increases and the relation between ζ and k is given in the table 1 below:

Table 1. *Solution of differential equation of cosine form*

$\zeta=$	0,05	0,1	0,15	0,2	0,25	0,3	0,35	0,4	0,45	0,5
k=	5,43	2,31	1,3	0,82	0,55	0,38	0,26	0,19	0,13	0,09

The equation (29) could take the form of a sinusoidal solution of the form (with appropriate choice of the initial conditions):

$$z(t) = \frac{\alpha V}{\omega_n \sqrt{1-\zeta^2}} \cdot e^{-\zeta \omega_n t} \cdot \sin\left(\omega_n \sqrt{1-\zeta^2} \cdot t\right) \qquad (31)$$

In this solution the relation between ζ and k is (the arc is equal to $\pi/2$):

$$k = \frac{e^{-\zeta \frac{\frac{\pi}{2}}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \qquad (32)$$

Table 2. *Solution of differential equation of sine form*

$\zeta=$	0,05	0,1	0,15	0,2	0,25	0,3	0,35	0,4	0,45	0,5
k=	0,93	0,86	0,80	0,74	0,69	0,64	0,59	0,55	0,51	0,46

From both Tables 1 and 2 we derive the necessity that the joints should be laid on a very soft support (low total static stiffness coefficient of the track) and simultaneously that they should present an increased damping coefficient.

4. Investigation of coefficients k_i

After almost twenty years of research in the Greek State Railways (OSE), the author ([3]) suggested that the equation (30), should be written (for a probability of occurrence of 68,3%:

$$Q_{dynamic} \approx \frac{k'_\alpha}{\underbrace{200 \cdot \sqrt{1,7804 \cdot 7,5}}_{k_\alpha}} \cdot V \cdot \sqrt{m_{NSM} \cdot h_{TRACK}} \qquad (33)$$

where k'_α should be verified for a great variety of lines: for newly ground rail-head to non-ground railhead in lines with speeds over $140 km/h$ or even for secondary lines with very low speeds ([18]).

In French literature [19], a value is given for the product k_α of the equation (30) of the present paper, as it is derived from measurements on track. J. Alias gives $\alpha = 2 \cdot 10^{-6}$ and for a track "already old/déjà ancienne" the equation (30) is transformed:

$$Q_{dynamic} = 1 \cdot \sigma(\Delta Q) \approx \frac{0,4444}{100} \cdot V \cdot \sqrt{m_{NSM} \cdot h_{TRACK}} \qquad (34)$$

where: V in $[km/h]$, h in $[t/mm]$, m_{NSM} in $[t]$ and $\sigma(\Delta Q)$ in $[t]$.
This implies that:

$$\frac{0,4444}{100} = k \cdot 2 \cdot 10^{-6} \Rightarrow k = \frac{0,4444}{2 \cdot 10^{-8}} \tag{35}$$

It is easy to calculate that the values of k'_α are given in the Table 3 below:

Table 3. *Values of k'_α*

	Ground Rail	Non Ground Rail
k'_α	$0,9 \div 1,8$	$1,8 \div 3,6$

For the case of secondary lines with very low speeds it could be even 7,5.
We could also approach the coefficient k_α of equation (33) given by:

$$k_\alpha = \frac{k'_\alpha}{200 \cdot \sqrt{1,7804 \cdot 7,5}} \tag{36}$$

Table 4. *Values of k_α*

	Ground Rail	Non Ground Rail
k_α	$3,894 \cdot 10^{-4} \div 7,788 \cdot 10^{-4}$	$7,788 \cdot 10^{-4} \div 15,577 \cdot 10^{-4}$

5. Conclusion

In the present paper the dynamic component of the Load and the reaction on each support point of the rail (sleeper) are investigated through the second order differential equation of motion of the Non Suspended Masses of the Vehicle and specifically the transient response of the reaction/action on each support point of the rail. The case of a deformed or bent joint or welding is analyzed through the second order differential equation of motion and the solution is investigated. The necessity, that the joints should be laid on a very soft support (low total static stiffness coefficient of the track) and simultaneously that they should present an increased damping coefficient, is derived by the analysis.

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