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RAILWAY TRACK: THE INFLUENCE OF LONG-WAVELENGTH DEFECTS ON THE ACTING LOADS AND TRACK DEFLECTION

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Abstract

The rail running table is a wave in space which is not perfectly "rectilinear", that is, it does not form a perfectly straight line but contains defects/ faults, varying from a few millimeters to several meters, and imposes forced oscillation on the railway vehicles that move on it. The defects with long wavelength, which play a key role on track deflection and the loads that develop, are analyzed using the second order differential equation of motion. The dynamic stiffness coefficient of the track, as well as the ratio between the period of the acting load (forcing period) compared to the eigenperiod of the track are of great importance, for the forces that are developed on track. The total static stiffness coefficient of the track, for the most common combination of track layers in high speed lines, as well as the relation of these two periods, as derived from a sensitivity analysis by variation of parameters, are analyzed in the present paper.

Keywords: railway track; dynamic stiffness; actions/ loads; deflection; subsidence; eigenperiod; forcing period.

1. INTRODUCTION

The railway track is usually modeled as a continuous beam on elastic support. Train circulation is a random dynamic phenomenon and, depending on the different frequencies of the loads it imposes, there is a corresponding response of the track superstructure. At the instant when an axle passes from the location of a sleeper, a random dynamic load is applied on the sleeper. The theoretical approach for the estimation of the dynamic loading of a sleeper requires the analysis of the total load acting on the sleeper to individual component loads-actions, which, in general, can be divided into:

- the static component of the load, and the relevant reaction/action per support point of the rail (sleeper)
- the dynamic component of the load, and the relevant reaction/action per support point of the rail (sleeper)

The static component of the load on a sleeper, in the classical sense, refers to the load undertaken by the sleeper when a vehicle axle at standstill is situated exactly on top of the sleeper. For dynamic loads with low frequencies the load is essentially static. The static load is further analyzed into individual component loads: the static reaction/action on a sleeper due to wheel load and the semi-static reaction/action due to cant deficiency [1].

The dynamic component of the load of the track depends on the mechanical properties (stiffness, damping) of the system "vehicle-track", and on the excitation caused by the vehicle's motion on the track (Figure 1). The response of the track to the aforementioned excitation results in the increase of the static loads on the superstructure. The dynamic load is primarily caused by the motion of the vehicle's Non-Suspended (Unsprung) Masses, which are excited by track geometry defects, and, to a smaller degree, by the effect of the Suspended (sprung) Masses. In order to formulate the theoretical equations for the calculation of the dynamic component of the load, the statistical probability of exceeding the calculated load -in real conditions- should be considered, so that the corresponding equations would refer to the standard deviation (variance) of the load [1, 2].

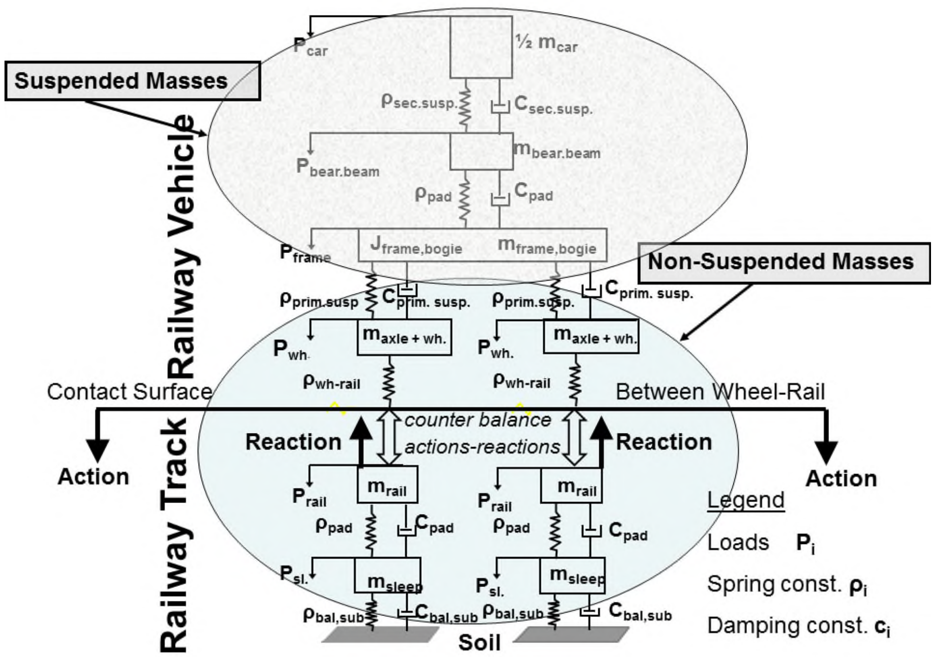

Figure 1: "Railway Vehicle - Railway Track" as an Ensemble of Springs and Dashpots.

In the present paper the dynamic component of the acting loads is investigated through the second order differential equation of motion of the Non Suspended Masses of the Vehicle and specifically the transient response of the reaction/ action on each support point (sleeper) of the rail.

2. CALCULATION OF THE ACTIONS ON EACH SUPPORT POINT OF RAIL

2.1. Static Action/Reaction

The most widely used theory (referred to as the Zimmermann theory or formula [3]) based on Winkler analysis [4] examines the track as a continuous beam on elastic support whose behavior is governed by the following equation [5]:

$$\frac{d^4 y}{dx^4} = -\frac{1}{E \cdot J} \cdot \frac{d^2 M}{dx^2} \tag{1}$$

where y is the deflection of the rail, M is the bending moment, J is the moment of inertia of the rail, and E is the modulus of elasticity of the rail.

From the formula above it is derived that the reaction of a sleeper R_{static} is:

$$R_{stat} = \frac{Q_{wheel}}{2\sqrt{2}} \cdot \sqrt[4]{\frac{\ell^3 \cdot \rho}{E \cdot J}} \Rightarrow \frac{R_{stat}}{Q_{wheel}} = \bar{A} = \bar{A}_{stat} = \frac{1}{2\sqrt{2}} \cdot \sqrt[4]{\frac{\ell^3 \cdot \rho}{E \cdot J}} \tag{2}$$

where Q_{wheel} the static wheel load, ℓ the distance among the sleepers, E and J the modulus of elasticity and the moment of inertia of the rail, R_{stat} the static reaction/action on the sleeper, and ρ reaction coefficient of the sleeper which is defined as: ρ=R/y, and is a quasi-coefficient of the track elasticity (stiffness) or a spring constant of the track. $\bar{A}=\bar{A}_{stat}$ equals to R_{stat}/Q_{wheel}, which is the percentage of the acting (static) load of the wheel that the sleeper undertakes as (static) reaction.

In reality, the track consists of a sequence of materials –in the vertical axis– (substructure, ballast, sleeper, elastic pad/ fastening, rail), that are characterized by their individual coefficients of elasticity (static stiffness coefficients) ρ_i (Figure 2). Hence, for each material:

$$\rho_i = \frac{R}{y_i} \Rightarrow y_i = \frac{R}{\rho_i} \Rightarrow y_{total} = \sum_{i=1}^{\nu} y_i = \sum_{i=1}^{\nu} \frac{R}{\rho_i} = R \cdot \sum_{i=1}^{\nu} \frac{1}{\rho_i} \Rightarrow \frac{1}{\rho_{total}} = \sum_{i=1}^{\nu} \frac{1}{\rho_i} \tag{3}$$

where ν is the number of various layers of materials that exist under the rail -including rail– elastic pad, sleeper, ballast etc.

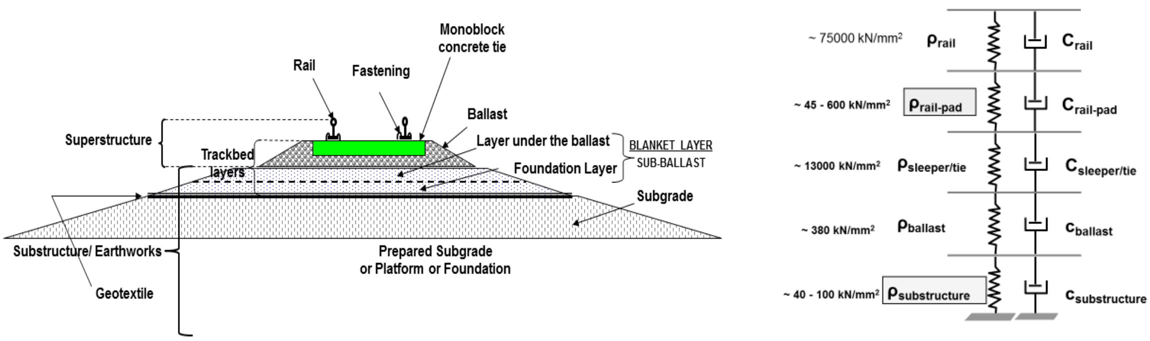

Figure 2: Cross-section of Ballasted Track and Characteristic Values of the Static Stiffness Coefficients.

2.2. Semi-static Action/ Reaction

The semi-static Action/Reaction is produced by the centrifugal acceleration exerted on the wheels of a vehicle that is running in a curve with cant deficiency. Cant deficiency or unbalanced superelevation [6, p. 604] is defined as the difference (deficit or excess in mm) of the designed superelevation in a curve from the theoretic one that is needed to fully counterbalance the centrifugal acceleration in the cross section of a track on a curve. It is not, however, a dynamic load in the sense of the load referred to in the next paragraph. Therefore, it is often considered to be a semi-static load. The following equation [1, 7, 8]:

$$Q_\alpha = \frac{2 \cdot \alpha \cdot h_{CG}}{e^2} \cdot Q_{wheel} \tag{4}$$

provides the increase Q_α of the vertical static load Q_{wheel} of the wheel, at curves with cant deficiency. In the above equation α is the cant deficiency, h_{CG} the height of the center of gravity of the vehicle from the rail head and e the track gauge. The semi-static reaction of the sleeper is:

$$R_{semi-stat} = \overline{A} \cdot Q_\alpha \tag{5}$$

and the total static reaction is:

$$R_{stat-total} = R_{stat} + R_{semi-stat} \tag{6}$$

2.3. General Solution of the Second Order Differential Equation of Motion for the Dynamic Component

The dynamic component of the acting load consists of the action due to the Suspended Masses (SM) and the action due to the Non Suspended Masses (NSM) of the vehicle. To the latter a section of the track mass is added, that participates in its motion [9]. The Suspended (sprung) Masses of the vehicle –masses situated above the primary suspension (Figure 1)– apply forces with very small influence on the trajectory of the wheel and on the excitation of the system. This enables the simulation of the track as an elastic media with damping as shown in Figure 2 which depicts the rolling wheel on the rail running table ([10], [11], [12], [13]). Forced oscillation is caused by the irregularities of the rail running table (simulated by an input random signal) –which are represented by n–, in a gravitational field with acceleration g. There are two suspensions on the vehicle for passenger comfort purposes: primary and secondary suspension. Moreover, a section of the mass of the railway track participates in the motion of the Non-Suspended (Unsprung) Masses of the vehicle. These Masses are situated under the primary suspension of the vehicle.

We approach the matter considering that the rail running table contains a longitudinal fault/ defect of the rail surface. In the above equation, the oscillation of the axle is damped after its passage over the defect. Viscous damping, due to the ballast, enters the above equation under the condition that it is proportional to the variation of the deflection dy/dt. To simplify the investigation, if the track mass (for its calculation see [9], [14]) is ignored -in relation to the much larger Vehicle's Non Suspended Mass- and bearing in mind that y+n is the total subsidence of the wheel during its motion (since the y and n are added algebraically), we can approach the problem of the random excitation, based on a cosine defect ($V \ll V_{critical}$=500 km/h):

$$\eta = a \cdot \cos \omega t = a \cdot \cos\left(2\pi \cdot \frac{V \cdot t}{\lambda}\right) \tag{7}$$

Where V the speed of the vehicle, T=2π/ω→ →ωt=2π/(Tt)=2πVt/λ where λ the length of the defect, run by the wheel in:

$$T = \frac{\lambda}{V} \Rightarrow \lambda = T \cdot V \tag{8}$$

If we set:

$$y = z + \frac{m_{SM} + m_{NSM}}{h_{TRACK}} \cdot g \Rightarrow \frac{dy}{dt} = \frac{dz}{dt} \quad and \quad \frac{d^2y}{dt^2} = \frac{d^2z}{dt^2} \tag{9}$$

where, the quantity $\frac{m_{SM}+m_{NSM}}{h_{TRACK}}\cdot g$ represents the subsidence due to the static loads only, and z random (see [12]) due to the dynamic loads, the second order differential equation of motion becomes:

$$m_{NSM}\frac{d^2z}{dt^2}+\Gamma\cdot\frac{dz}{dt}+h_{TRACK}\cdot z=-m_{NSM}\cdot\frac{d^2n}{dt^2}\Rightarrow \tag{10a}$$

$$\Rightarrow m_{NSM}\left(\frac{d^2z}{dt^2}+\frac{d^2n}{dt^2}\right)+\Gamma\cdot\frac{dz}{dt}+h_{TRACK}\cdot z=0 \tag{10b}$$

By setting ω=2πV/ℓ and considering defects of sinusoidal form:

$$n=a\cdot\cos(\omega t)=a\cdot\cos\left(2\pi\frac{Vt}{\lambda}\right)$$

where λ is the wavelength of the defect, run through by the wheel, and λ=T·V, then:

$$\omega_n^2=\frac{h_{TRACK}}{m_{NSM}},\quad h_{TRACK}=2\sqrt{2}\cdot\sqrt[4]{\frac{E\cdot J\cdot\rho_{total}^3}{\ell^3}} \tag{11}$$

where ℓ distance among the sleepers. Equations (10) give:

$$m_{NSM}\frac{d^2z}{dt^2}+\Gamma\cdot\frac{dz}{dt}+h_{TRACK}\cdot z=-m_{NSM}\cdot a\cdot\omega^2\cdot\cos(\omega t) \tag{12}$$

The complete solution of which using polar coordinates is ([5], p.199 and ch.3):

$$z=\underbrace{A\cdot e^{-\zeta\omega_n t}\cdot\sin\left(\omega_n t\sqrt{1-\zeta^2}-\varphi\right)}_{transient-part}+\underbrace{a\cdot B\cdot\cos(\omega t-\varphi)}_{steady-state-part} \tag{13}$$

where the first term is the transient part and the second part is the steady state.

3. INVESTIGATION OF THE TRANSIENT AND STEADY STATE TERM OF MOTIONS

3.1. Solution of the Second Order Differential Equation of Motion for a Cosine Defect

We focus herein on the term $A\cdot e^{-\zeta\omega_n t}\cdot\sin\left(\omega_n t\sqrt{1-\zeta^2}-\varphi\right)$ from Equation 13 which represents the transient part of motion. We investigate this term for ζ=0.

The theoretical analysis for the additional –to the static and semi-static component– dynamic component of the load due to the Non Suspended Masses and the Suspended Masses of the vehicle, leads to the examination of the influence of the Non Suspended Masses only, since the frequency of oscillation of the Suspended Masses is much smaller than the frequency of the Non Suspended Masses. If m_{NSM} represents the Non Suspended Mass, m_{SM} the Suspended Mass and m_{TRACK} the Track Mass participating in the motion of the Non Suspended Masses of the vehicle, the differential equation is (with no damping *ζ=0*):

$$m_{NSM}\cdot\frac{d^2z}{dt^2}+h_{TRACK}\cdot z=m_{NSM}\cdot g\Rightarrow \tag{14a}$$

$$(m_{NSM}+m_{TRACK})\cdot\frac{d^2z}{dt^2}+h_{TRACK}\cdot z=m_{NSM}\cdot g \tag{14b}$$

Where: g the acceleration of gravity and

$$h_{TRACK}=2\sqrt{2}\cdot\sqrt[4]{\frac{EJ\rho_{total}^3}{\ell^3}} \tag{15}$$

ρ_{total} the total static stiffness coefficient of the track, ℓ the distance among the sleepers, E, J the modulus of elasticity and the moment of inertia of the rail.

The theoretic calculation of m_{TRACK} gives as result:

$$m_{TRACK} = 2\sqrt{2} \cdot m_0 \cdot \sqrt[4]{\frac{EJ\ell}{\rho_{total}}} \tag{16}$$

For a comparison of the theoretical track mass to measurement results refer to [13] and [14]. The particular solution of the differential Equation (14b) corresponds to the static action of the weight of the wheel:

$$z = \frac{m_{TRACK} \cdot g}{h_{TRACK}}$$

We assume that the rolling wheel runs over an isolated sinusoidal defect of length λ of the form:

$$n = \frac{a}{2} \cdot \left(1 - \cos\frac{2\pi x}{\lambda}\right) = \frac{a}{2} \cdot \left(1 - \cos\frac{2\pi Vt}{\lambda}\right) \tag{17}$$

where n is the ordinate of the defect. Consequently, the ordinate of the center of inertia of the wheel is n+z. Defining τ_1 as the time needed for the wheel to pass over the defect at a speed V:

$$\tau_1 = \frac{\lambda}{V} \tag{18}$$

the differential equation of the motion of the wheel is:

$$m_{NSM} \cdot \frac{d^2}{dt^2}(z+n) + m_{TRACK} \cdot \frac{d^2 z}{dt^2} + h_{TRACK} \cdot z = 0 \Rightarrow$$

$$(m_{NSM} + m_{TRACK}) \cdot \frac{d^2 z}{dt^2} + h_{TRACK} \cdot z = -m_{NSM} \cdot \frac{d^2 n}{dt^2} = -m_{NSM} \cdot \frac{2a\pi^2}{\tau_1^2} \cdot \cos\frac{2\pi t}{\tau_1} \tag{19}$$

Since:

$$\frac{dn}{dt} = \frac{a}{2} \cdot \frac{2\pi V}{\lambda} \cdot \sin\frac{2\pi Vt}{\lambda} = \frac{a}{2} \cdot \frac{2\pi\lambda}{\lambda \cdot \tau_1} \cdot \sin\frac{2\pi Vt}{\lambda} = \frac{a}{2} \cdot \frac{2\pi}{\tau_1} \cdot \sin\frac{2\pi Vt}{\lambda} \Rightarrow \tag{20}$$

$$\frac{d^2 n}{dt^2} = -\frac{a}{2} \cdot \left(\frac{2\pi}{\tau_1}\right)^2 \cdot \cos\frac{2\pi Vt}{\lambda} = -\frac{a}{2} \cdot \left(\frac{2\pi}{\tau_1}\right)^2 \cdot \cos\frac{2\pi\lambda t}{\lambda \cdot \tau_1} = -\frac{2a\pi^2}{\tau_1^2} \cdot \cos\frac{2\pi t}{\tau_1} \tag{21}$$

$$x = V \cdot t, \quad \omega_1 = 2 \cdot \frac{\pi \cdot V}{\ell} = \frac{2\pi}{T}, \quad \omega_n^2 = \frac{h_{TRACK}}{m_{NSM}} \tag{22}$$

where ω_1 the cyclic frequency of the external force and ω_n the natural frequency.

The additional dynamic component of the load due to the motion of the wheel is:

$$-m_{NSM} \cdot (z'' + n'') = h_{TRACK} \cdot z + m_{TRACK} \cdot z'' \tag{23}$$

To solve Equation (19) we divide by (m_{NSM}+m_{TRACK}):

$$\frac{d^2 z}{dt^2} + \frac{h_{TRACK}}{(m_{NSM} + m_{TRACK})} \cdot z = -\frac{m_{NSM}}{(m_{NSM} + m_{TRACK})} \cdot \frac{2a\pi^2}{\tau_1^2} \cdot \cos\frac{2\pi t}{\tau_1} \tag{24}$$

Equation (24) is the typical second order differential equation of motion for an undamped forced harmonic motion, of the form [15] :

$$m \cdot \ddot{z} + kz = p_0 \cos(\omega_1 t) \Rightarrow \ddot{z} + \frac{k}{m} z = \frac{p_0}{m} \cos(\omega_1 t) = \omega_n^2 \cdot \frac{p_0}{k} \cos(\omega_1 t), \quad \omega_n^2 = \frac{k}{m} \Rightarrow m = \frac{k}{\omega_n^2} \tag{25}$$

The complete solution is (see Annex 1):

$$z(t) = \frac{p_0}{k} \cdot \frac{1}{1 - \left(\frac{\omega_1}{\omega_n}\right)^2} \cdot \left[\underbrace{\cos(\omega_1 t)}_{steady-state} - \underbrace{\cos(\omega_n t)}_{transient-part}\right] \tag{26}$$

3.2 Application for a Railway Track

Equation (24) can be equated to equation (25), when:

k=h_{TRACK}, m=m_{NSM}+m_{TRACK}, and

$$\omega_n^2 = \frac{h_{TRACK}}{m_{NSM} + m_{TRACK}}, \quad p_0 = -\frac{2 \cdot \alpha \cdot \pi^2 \cdot m_{NSM}}{\tau_1^2} \tag{27}$$

The general solution of equation (24) –derived from equation (26)- is:

$$z(t) = -\underbrace{\frac{2\cdot\alpha\cdot\pi^2\cdot m_{NSM}}{\tau_1^2}}_{p_0}\cdot\frac{1}{h_{TRACK}}\cdot\frac{1}{1-\left(\frac{\omega_1}{\omega_n}\right)^2}\cdot\left[\underbrace{\cos(\omega_1 t)}_{steady-state}-\underbrace{\cos(\omega_n t)}_{transient-part}\right]\Rightarrow$$

$$z(t) = -\frac{1}{2}\cdot\underbrace{\frac{4\cdot\pi^2}{\tau_1^2}}_{\omega_1^2}\cdot\frac{\alpha\cdot m_{NSM}}{\underbrace{\omega_n^2\cdot(m_{NSM}+m_{TRACK})}_{h_{TRACK}}}\cdot\frac{1}{1-\left(\frac{\omega_1}{\omega_n}\right)^2}\cdot\left[\underbrace{\cos(\omega_1 t)}_{steady-state}-\underbrace{\cos(\omega_n t)}_{transient-part}\right] = -\frac{\alpha\cdot m_{NSM}\cdot\left(\frac{\omega_1}{\omega_n}\right)^2}{2(m_{NSM}+m_{TRACK})\cdot\left[1-\left(\frac{\omega_1}{\omega_n}\right)^2\right]}\cdot\left[\underbrace{\cos(\omega_1 t)}_{steady-state}-\underbrace{\cos(\omega_n t)}_{transient-part}\right]\Rightarrow$$

$$z(t) = \frac{\alpha}{2}\cdot\frac{m_{NSM}}{(m_{NSM}+m_{TRACK})}\cdot\frac{1}{1-\left(\frac{\omega_n}{\omega_1}\right)^2}\cdot\left[\underbrace{\cos(\omega_1 t)}_{steady-state}-\underbrace{\cos(\omega_n t)}_{transient-part}\right] \quad (28)$$

where, $T_n=2\pi/\omega_n$ the period of the free oscillation of the wheel circulating on the rail and $T_1=2\pi/\omega_1$ the necessary time for the wheel to run over a defect of wavelength λ: $T_1=\lambda/V$. Consequently, $T_n/T_1=\omega_1/\omega_n$.

From equation (28):

$$\frac{(m_{NSM}+m_{TRACK})}{m_{NSM}}\cdot z(t) = \alpha\cdot\frac{1}{2}\cdot\frac{1}{1-\left(\frac{\omega_n}{\omega_1}\right)^2}\cdot\left[\underbrace{\cos(\omega_1 t)}_{steady-state}-\underbrace{\cos(\omega_n t)}_{transient-part}\right] \quad (29)$$

We can investigate Equation (29) after a sensitivity analysis by variating parameters: for given values of $T_n/T_1=\omega_1/\omega_n$ and for given value of V (for example equal to 1) the time period T_1 is proportional to μ=0.1, 0.2, … 1.0 of defect λ. Equation (29) is transformed:

$$\left[\frac{(m_{NSM}+m_{TRACK})}{m_{NSM}}\cdot z(t)\cdot\frac{1}{\alpha}\right] = \frac{1}{2}\cdot\frac{1}{1-(n)^2}\cdot\left[\underbrace{\cos(\omega_1 t)}_{steady-state}-\underbrace{\cos(n\cdot\omega_1 t)}_{transient-part}\right] = \quad (29\ a)$$

where $n=\omega_n/\omega_1$. $\omega_1=\lambda/V$ and we examine values of $\mu\cdot\lambda$=0, 0.1λ, 0.2λ,...., 0.8λ, 0.9λ, λ.

$$Eq.(29a) = \frac{1}{2}\cdot\frac{1}{1-(n)^2}\cdot\left[\underbrace{\cos\left(\frac{2\pi V}{\lambda}\cdot\frac{\mu\cdot\lambda}{V}\right)}_{steady-state}-\underbrace{\cos\left(n\cdot\frac{2\pi V}{\lambda}\frac{\mu\cdot\lambda}{V}\right)}_{transient-part}\right] = \frac{1}{2}\cdot\frac{1}{1-(n)^2}\cdot\left[\underbrace{\cos(2\pi\cdot\mu)}_{steady-state}-\underbrace{\cos(n\cdot 2\pi\cdot\mu)}_{transient-part}\right] \quad (29b)$$

for discrete values of $n=\omega_n/\omega_1$ ($=T_1/T_n$) and μ a percentage of the wavelength λ. In Figure 4 Equations (29a and b) are depicted. The first term in the bracket of Equation (29a) is depicted on the vertical axis while on the horizontal axis the percentages of the wavelength $\mu\cdot\lambda$ are shown.

We observe that z(x) has its maximum value for T_1/T_n=0,666667=2/3, equal to 1,465:

$$z(t) = \left[\frac{m_{NSM}}{(m_{NSM}+m_{TRACK})}\right]\cdot\alpha\cdot 1{,}465\text{, for x=0,91}\lambda \quad (29c)$$

The relation T_1/T_n represents the cases for short and long wavelength of the defects. For T_1/T_n=2-2,5 the wavelength is long and for T_1/T_n the wavelength is short ([8], p.49). The second derivative of z(x) from equation (28), that is the vertical acceleration that gives the dynamic overloading due to the defect, is calculated:

$$z'(t) = \frac{\alpha}{2}\cdot\frac{m_{NSM}}{(m_{NSM}+m_{TRACK})}\cdot\frac{1}{1-\left(\frac{\omega_n}{\omega_1}\right)^2}\cdot\left[\underbrace{-\omega_1\cdot\sin(\omega_1 t)}_{steady-state}+\underbrace{\omega_n\cdot\sin(\omega_n t)}_{transient-part}\right] \quad (30)$$

$$z''(t) = -\frac{\alpha}{2}\cdot\frac{m_{NSM}}{(m_{NSM}+m_{TRACK})}\cdot\frac{1}{1-\left(\frac{\omega_n}{\omega_1}\right)^2}\cdot\left[\underbrace{\omega_1^2\cdot\cos(\omega_1 t)}_{steady-state}-\underbrace{\omega_n^2\cdot\cos(\omega_n t)}_{transient-part}\right] \quad (31)$$

for discrete values of $n=\omega_n/\omega_1$ ($=T_1/T_n$) and μ a percentage of the wavelength λ, and T_n=0,0307 sec as calculated above. The additional subsidence of the deflection z at the beginning of the defect is negative in the first part of the defect. Following the wheel's motion, z turns to positive sign and reaches its maximum and possibly afterwards z becomes again negative. After the passage of the wheel over the defect, one oscillation occurs which approaches to the natural cyclic frequency ω_n (this oscillation is damped due to non-existence of a new defect since we considered one isolated defect) in reality, even if in the present analysis the damping was omitted for simplicity. The maximum value of z is given in Table 1 below, as it is –graphically– measured in Figure 3.

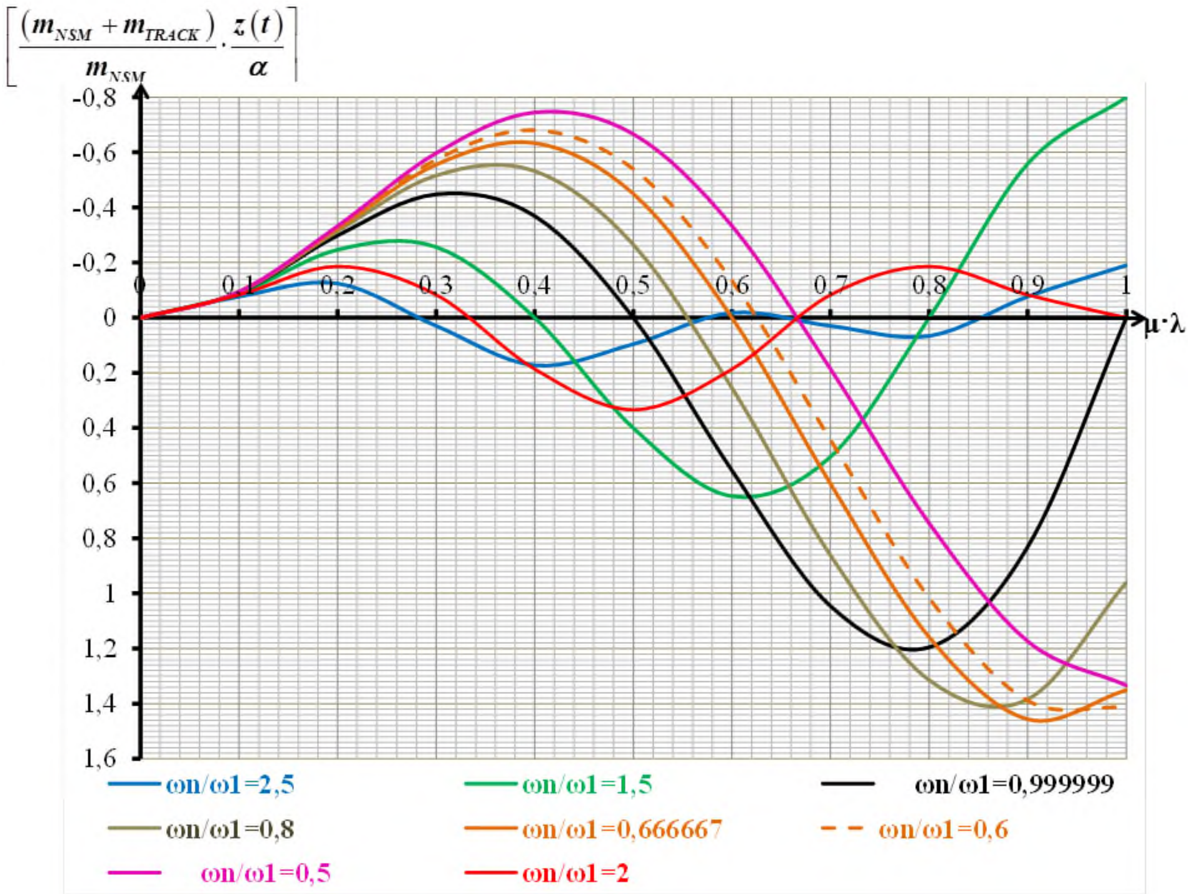

Figure 3: Mapping of Equation (29b). On the Horizontal Axis the percentage of the wavelength λ of the defect is depicted. On the Vertical Axis the first term of equation (29a), inside the brackets, is depicted.

Table 1: Maximum Values of $\zeta=[(m_{NSM}+m_{TRACK})/m_{NSM}]\cdot[z_{max}/\alpha]$

T_1/T_n	2,5	2	1,5	1	0,8	0,6667	0,6	0,5
ζ	0,18	0,335	0,65	1,205	1,415	1,47	1,43	1,34
where: $\zeta=[(m_{NSM}+m_{TRACK})/m_{NSM}]\cdot[z_{max}/\alpha]$								

It is observed that the maximum value is shifted towards the end of the defect as the ratio T_1/T_n decreases, that is when the defect's wavelength becomes short. The maximum is obtained for T_1/T_n = 0,666667 = 2/3. For each combination of "*vehicle + track section*" the critical value of the speed V, for which the 2/3 are achieved is a function of the wavelength λ. Since:

$$T_1=\frac{\lambda}{V}\Rightarrow V=\frac{\lambda}{T_1}=\frac{3}{2}\cdot\frac{\lambda}{T_n} \qquad (32)$$

We can calculate the critical V for any combination of track layers and their corresponding stiffnesses. As an example we use the ballasted track depicted in Figure 2, equipped with rail UIC60 (ρ_{rail}=75.000 kN/mm), monoblock sleepers of prestressed concrete B70 type ($\rho_{sleeper}$=13.500 kN/mm), W14 fastenings combined with pad Zw700 Saargummi (ρ_{pad} fluctuating from 50,72 to 48,52 kN/mm), ballast fouled after 2 years in circulation ($\rho_{ballast}$=380 kN/mm) and subgrade/substructure fluctuating from $\rho_{subgrade}$=250 kN/mm for rocky tunnel bottom or concrete bridge with 30 cm ballast on it to $\rho_{subgrade}$=40 kN/mm for pebbly substructure/subgrade). For this cross section of ballasted track, h_{TRACK} ranges between 100,71 kN/mm = 10.071 t/m and 64,83 kN/mm = 6.483 t/m and m_{TRACK} ranges from 0,409 t to 0,457 t (for the calculations see [9], [14]). If we consider an average m_{NSM}=1,5 t, then:

$$m_{NSM}+m_{TRACK}=\frac{1,5+0,433}{9,81}=0,197 \quad tons-mass \tag{33}$$

Where g = 9,81 m/sec², the acceleration of gravity and an average m_{TRACK} = 0,433 t. The period T_n is given by:

$$T_n=2\pi\sqrt{\frac{1,5+0,433}{9,81\cdot 8277}}=0,0307\ \sec\Rightarrow V_{critical}=\frac{3}{2}\cdot\frac{\lambda}{T_n}=\frac{3}{2}\cdot\frac{\lambda}{0,0307}=48,86\cdot\lambda \tag{34}$$

where an average h_{TRACK}=8277 t/m is used and V_c is given in [m/sec], λ in [m]. For a wavelength of 1,2 m, V_c=58,63 m/sec=211 km/h.

3.3. Defects of Long Wavelength

If we consider a defect with a wavelength that produces a forced oscillation with $\frac{T_1}{T_n}=\frac{\omega_n}{\omega_1}=2,5$, we calculate (from Figure 4 maximum value is 0,19, for x=0,41·λ):

$$z_{\max}=\left[\frac{m_{NSM}}{(m_{NSM}+m_{TRACK})}\right]\cdot\alpha\cdot 0,19=0,147\cdot\alpha \tag{35}$$

With the values calculated above: T_n=0,0307 sec, T_1=0,07675 sec, the wavelength ℓ equals to (from Equation (32)):

$$\lambda=V\cdot T_1=2,5\cdot V\cdot T_n=0,07675\cdot 58,63=4,5m \tag{36}$$

This value represents a defect of long wavelength.

The static deflection due to a wheel load of 11,25 t or 112,5 kN is equal to:

$$z_{static}=\frac{Q_{wheel}}{2\sqrt{2}}\cdot\sqrt[4]{\frac{\ell^3}{EJ\rho^3_{total}}}=\frac{112.500N}{2\sqrt{2}}\cdot\sqrt[4]{\frac{600^3 mm^3}{210.000\frac{N}{mm^2}\cdot 3,06\cdot 10^7 mm^4\cdot 82.770^3\frac{N^3}{mm^3}}}=\frac{112.500N}{2\sqrt{2}}\cdot 1,560355\cdot 10^{-5}\frac{mm}{N}=0,62mm$$

Consequently, for α=1 mm, that is for every mm of vertical defect, the dynamic increment of the static deflection is equal to (0,147/0,62)=23,7% of the static deflection (for every mm of the depth of the defect).

If we examine the second derivative (vertical acceleration) as a percentage of g, the acceleration of gravity, then [from equation (31)]:

$$\left[\frac{(m_{NSM}+m_{TRACK})}{m_{NSM}}\cdot\frac{z''(t)}{\alpha}\right]=-\frac{1}{2}\cdot\frac{\omega_1^2}{1-\left(\frac{\omega_n}{\omega_1}\right)^2}\cdot\left[\underbrace{\cos\left(\frac{2\pi V}{\lambda}\cdot\frac{\mu\cdot\lambda}{V}\right)}_{steady-state}-\underbrace{\frac{\omega_n^2}{\omega_1^2}\cdot\cos\left(n\cdot\frac{2\pi V}{\lambda}\frac{\mu\cdot\lambda}{V}\right)}_{transient-part}\right]\Rightarrow \tag{37a}$$

$$\left[\frac{(m_{NSM}+m_{TRACK})}{m_{NSM}}\cdot\frac{z''(t)}{\alpha}\right]=-\frac{1}{2}\cdot\frac{\left(\frac{2\pi}{n\cdot T_n}\right)^2}{1-(n)^2}\cdot\left[\underbrace{\cos(2\pi\mu)}_{steady-state}-\underbrace{(n)^2\cdot\cos(2n\pi\mu)}_{transient-part}\right]$$

$$\left[\frac{(m_{NSM}+m_{TRACK})}{m_{NSM}}\cdot\frac{1}{g}\cdot\frac{z''(t)}{\alpha}\right]=-\left(\frac{2\pi}{n\cdot T_n}\right)^2\cdot\frac{1}{g}\cdot\left[\frac{1}{2\cdot\left[1-(n)^2\right]}\cdot\left[\underbrace{\cos(2\pi\mu)}_{steady-state}-\underbrace{(n)^2\cdot\cos(2n\pi\mu)}_{transient-part}\right]\right]\quad[\%g] \tag{37b}$$

Equation (37b) is plotted in Figure 5 . The first term in the bracket of Equation (37b) is depicted on the vertical axis while on the horizontal axis the percentages of the wavelength μ·λ are shown.

For the case calculated above in Figure 5, at the point *x=0,41·λ* the term in bracket has a value of -0.332:

$$\left[\frac{(m_{NSM}+m_{TRACK})}{m_{NSM}}\cdot\frac{1}{g}\cdot\frac{z''(t)}{\alpha}\right]=-0.332041\Rightarrow z''(t)=-0.332041\cdot\frac{1,5}{1,5+0,4333}\cdot g\cdot\alpha=0,258\cdot g\cdot\alpha \tag{38}$$

Equation (14a) (its second part corresponds to the static action of the wheel load) has as particular solution: $z = \frac{m_{NSM} \cdot g}{h_{TRACK}}$. Abandoning the second part leads to the classic solution where z is the supplementary subsidence owed to the dynamic increase of the Load.

The dynamic increase of the Load is equal to:

$$Q_{dynamic} = h_{TRACK} \cdot z + m_{TRACK} \cdot z'' = 82770 \cdot 0{,}147 - 433kg \cdot 0{,}258 \cdot 9{,}81 \frac{m}{\sec^2} = 11071N = 1{,}1t \qquad (39)$$

Where, from the analysis above: h_{TRACK}=8277 t/m=82770 N/mm, m_{TRACK}=0,433 t=433 kg.

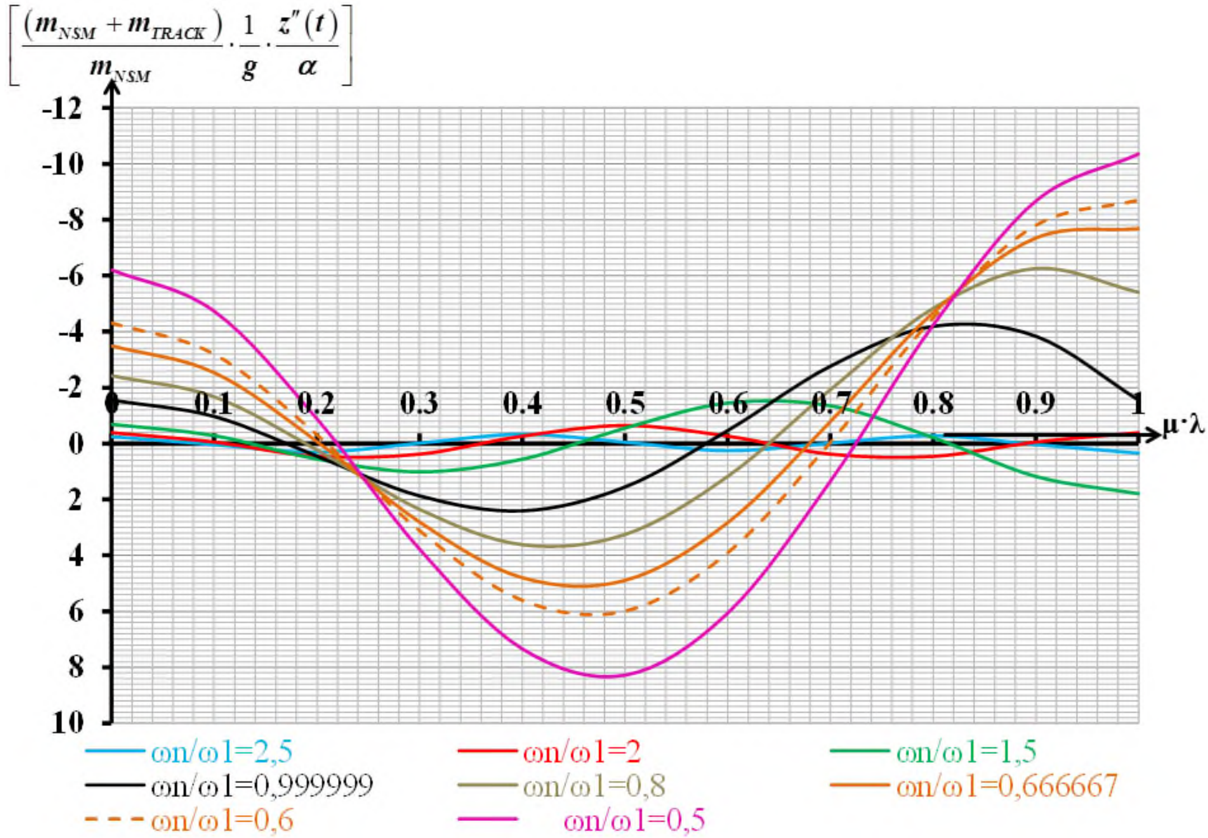

Figure 5: Mapping of the equation (37b), for the vertical acceleration due to a defect of long wavelength. In the Horizontal Axis the percentage of the wavelength λ of the defect is depicted. In the Vertical Axis the first term of equation (37a), in the brackets, is depicted.

Consequently, for arc height (i.e. sagitta) α=1 mm of a defect of wavelength λ, that is for every mm of vertical defect, the dynamic increase of the load is equal to (1,1/11,25)=9,78% of the static load of the wheel (for every mm of the depth of the defect).

4. CONCLUSIONS

For a defect of wavelength λ and sagitta of 1 mm (depth of the defect), the dynamic increase of the acting load –compared to the static wheel load– is equal to 9,78%. Furthermore from Figures 4 and 5, it is verified that when the speed increases, the period T1 decreases and the supplementary sagitta (depth of the defect) increases. The increase of the dynamic component of the load increases faster since it is dependent on the square of the speed $(\omega_1)^2$. When the dynamic stiffness coefficient h_{TRACK} increases, T_n decreases, T_1/T_n increases and the supplementary sagitta decreases for the same speed.

ANNEX 1

For the free oscillation (without external force) the equation is:

$$m\cdot\ddot{z}+k\cdot z=0\Rightarrow\ddot{z}+\frac{k}{m}\cdot z=0\Rightarrow\ddot{z}+\omega_n^2\cdot z=0 \quad (1.1.)$$

The general solution is [4]: $z(t)=A\cdot\cos(\omega_n t)+B\cdot\sin(\omega_n t)=z(0)\cdot\cos(\omega_n t)+\frac{\dot{z}(0)}{\omega_n}\cdot\sin(\omega_n t)$ (1.2)

Where:

$$A=z(0),\quad B=\frac{\dot{z}(0)}{\omega_n} \quad (1.3)$$

If we pass to the undamped harmonic oscillation of the form:

$$m\cdot\ddot{z}+k\cdot z=p_0\cdot\cos(\omega t)\Rightarrow\ddot{z}+\omega_n^2\cdot z=\frac{p_0}{m}\cdot\cos(\omega t)\Rightarrow\ddot{z}+\omega_n^2\cdot z=\omega_n^2\cdot\frac{p_0}{k}\cdot\cos(\omega t) \quad (1.4)$$

where: $\omega_n^2=\frac{k}{m}\Rightarrow m=\frac{k}{\omega_n^2}$ (1.5)

The particular solution of the linear second order differential equation (1.4) is of the form:

$$z_p(t)=C\cdot\cos(\omega t)\Rightarrow\dot{z}_p(t)=-\omega\cdot C\cdot\sin(\omega t)\Rightarrow\ddot{z}_p(t)=-\omega^2\cdot C\cdot\cos(\omega t) \quad (1.6)$$

Substituting equation (1.6) to equation (1.4) we derive:

$$-\omega^2\cdot C\cdot\cos(\omega t)+\omega_n^2\cdot C\cdot\cos(\omega t)=\frac{p_0}{m}\cdot\cos(\omega t)\Rightarrow-\omega^2\cdot C+\omega_n^2\cdot C=\omega_n^2\cdot\frac{p_0}{k}\Rightarrow$$

$$C\left(\omega_n^2-\omega^2\right)=\omega_n^2\cdot\frac{p_0}{k}\Rightarrow C=\frac{\omega_n^2}{\left(\omega_n^2-\omega^2\right)}\cdot\frac{p_0}{k}\Rightarrow C=\frac{p_0}{k}\cdot\frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2} \quad (1.7)$$

<u>The general solution for the equation (1.4)</u> is the addition of the solution (1.2) and of the solution of the equation (1.6) combined with equation (1.7):

$$z(t)=A\cdot\cos(\omega_n t)+B\cdot\sin(\omega_n t)+\frac{p_0}{k}\cdot\frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2}\cdot\cos(\omega t) \quad (1.8)$$

We have to calculate A and B. The first derivative of equation (1.8) is:

$$\dot{z}(t)=-\omega_n\cdot A\cdot\sin(\omega_n t)+\omega_n\cdot B\cdot\cos(\omega_n t)-\frac{p_0}{k}\cdot\frac{\omega}{1-\left(\frac{\omega}{\omega_n}\right)^2}\cdot\sin(\omega t) \quad (1.9)$$

Calculating the values of equation (1.8) and (1.9) at t=0:

$$z(0)=A\cdot\cos(0)+B\cdot\sin(0)+\frac{p_0}{k}\cdot\frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2}\cdot\cos(0)\Rightarrow A=z(0)-\frac{p_0}{k}\cdot\frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2} \quad (1.10)$$

$$\dot{z}(0)=-\omega_n\cdot A\cdot\sin(0)+\omega_n\cdot B\cdot\cos(0)-\frac{p_0}{k}\cdot\frac{\omega}{1-\left(\frac{\omega}{\omega_n}\right)^2}\cdot\sin(0)\Rightarrow B=\frac{\dot{z}(0)}{\omega_n} \quad (1.11)$$

Substituting equation (1.10) and (1.11) to the equation (1.8) we derive:

$$z(t)=\underbrace{\left[z(0)-\frac{p_0}{k}\cdot\frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right]\cdot\cos(\omega_n t)+\frac{\dot{z}(0)}{\omega_n}\cdot\sin(\omega_n t)}_{transient-part}+\underbrace{\frac{p_0}{k}\cdot\frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2}\cdot\cos(\omega t)}_{steady-state} \quad (1.12)$$

and for initial conditions z(0)=ż(0)=0: $z(t)=\frac{p_0}{k}\cdot\frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2}\cdot\left[\underbrace{\cos(\omega t)}_{steady-state}-\underbrace{\cos(\omega_n t)}_{transient-part}\right]$ (1.13)

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